A Network Approach to Bayes-Nash Incentive Compatible Mechanisms

Rudolf Müller[1], Andrés Perea[2], Sascha Wolf[3]

[1] Maastricht University, Faculty of Economics and Business Administration,
Department of Quantitative Economics
6200 MD, Maastricht, P.O. Box 616, The Netherlands
r.muller@ke.unimaas.nl

[2] Maastricht University, Faculty of Economics and Business Administration,
Department of Quantitative Economics
6200 MD, Maastricht, P.O. Box 616, The Netherlands
a.perea@ke.unimaas.nl

[3] Maastricht University, Faculty of Economics and Business Administration,
Department of Quantitative Economics
6200 MD, Maastricht, P.O. Box 616, The Netherlands
s.wolf@ke.unimaas.nl

Abstract. This paper provides a characterization of Bayes-Nash incentive compatible mechanisms in settings where agents have one-dimensional or multi-dimensional types, quasi-linear utility functions and interdependent valuations. The characterization is derived in terms of conditions for the underlying allocation function.
We do this by making a link to network theory and building complete directed graphs for agents' type spaces. We show that an allocation rule is Bayes-Nash incentive compatible if and only if these graphs have no negative, finite cycles.
In the case of one-dimensional types and given certain properties for agents' valuation functions, we show that this condition reduces to the absence of negative 2-cycles. In the case of multi-dimensional types and given a linearity requirement on the valuation functions, we show that this condition reduces to the absence of negative 2-cycles and an integratebility condition on the valuation functions.

Keywords. Implementation, Mechanism Design, Multi-Dimensional Types

1 Introduction

This paper is concerned with the characterization of Bayes-Nash incentive compatible mechanisms in social choice settings where agents have independently distributed, one-dimensional or multi-dimensional types and quasi-linear utility functions, i.e. utility is the valuation of an allocation minus a payment. We allow for interdependent valuations across agents.

We consider direct revelation mechanisms (DRM). Direct mechanisms consist of two rules: an allocation rule and a payment rule. There are examples in

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the literature offering characterizations of incentive compatible mechanisms in terms of a monotonicity condition on the allocation function, see for example Bikhchandani, Chatterji & Sen [1], Gui, Müller & Vohra [2] for dominant strategy incentive compatibility and Myerson [3] for Bayes-Nash incentive compatibility.

Similar to Gui, Müller & Vohra [2] we are making a link to network theory. If an allocation rule is Bayes-Nash incentive compatible then an agent's expected utility for truthfully reporting type t is at least as high as his expected utility for misreporting type s. We also have that an agent's expected utility for truthfully reporting type s is at least as high as his expected utility for misreporting type t. Taking together these two conditions we get the so-called 2-cycle inequality. It expresses that the expected difference in valuation for truthfully reporting t instead of misreporting s should be at least as big as the expected difference in valuation for misreporting t instead of truthfully reporting s. That the 2-cycle inequality holds is a necessary condition for Bayes-Nash incentive compatibility.

Recognizing that the constraints inherent in the definition of Bayes-Nash incentive compatibility have a natural network interpretation we build complete directed graphs for agents' type spaces. To do so we associate a node with each type and put a directed edge between each ordered pair of nodes. The length of the edge going from the node associated with type s to the node associated with the type t is defined as the cost of manipulation, i.e. the expected difference in an agent's valuation for truthfully reporting t instead of misreporting s. That the 2-cycle inequality holds translates into the absence of negative 2-cycles in these graphs.

We show that an allocation function is Bayes-Nash incentive compatible if and only if these graphs have no negative, finite cycles. In the case of one-dimensional types and given certain properties for agents' valuation functions, we show that this condition reduces to the absence of negative 2-cycles. As an example we can consider the setup of Myerson [3] and derive as a corollary his characterization for Bayes-Nash incentive compatibility which requires monotonicity for the expected conditional probability that an agent gets the item.

In the case of multi-dimensional types and given a linearity condition on the valuation functions, we show that the absence of negative, finite cycles can be reduced to the absence of negative 2-cycles and an integratebility condition on the valuation function. As examples we can look at the single item case with externalities considered by Jehiel, Moldovanu & Stacchetti [4] and the multi-item case considered by Jehiel & Moldovanu [5] and derive their characterizations for Bayes-Nash incentive compatibility as corollaries.

Note that this preliminary draft only contains some results and their proofs. The final paper will include the aforementioned examples, some further results for the multi-dimensional setting and a more elaborate discussion of how the contents of this paper relate to the existing literature.

2 Notation

We have a set of agents $N = \{1, \dots, n\}$. Each agent i has a type $t^i \in T^i$ with $T^i \subseteq IR^k$. T denotes the set of all type profiles $t = \left(t^1, \dots, t^n\right)$, and T^{-i} denotes the set of all type profiles $t^{-i} = \left(t^1, \dots, t^{i-1}, t^{i+1}, \dots, t^n\right)$. A **payment rule** is a function

$$P : T \mapsto IR^n,$$

so given a report r^{-i} of the others, reporting a type r^i results in a payment $P_i\left(r^i, r^{-i}\right)$ for agent i. Denoting the set of outcomes by Γ, an **allocation rule** is a function

$$f : T \mapsto \Gamma.$$

We allow for interdependent valuations across agents, i.e. agents' valuations do not only depend on their own types but on the types of all agents. As an example one can think of an auction for a painting (see Klemperer [6]) where agents' types reflect how much they like the painting. Here an agent's valuation for owning the painting can depend on the types of the others as the possible resale value and the owner's prestige are affected by them.

Take agent i having true type t^i and reporting r^i while the others have true types t^{-i} and report r^{-i}. The value that agent i assigns to the resulting allocation is denoted by $v^i\left(f\left(r^i, r^{-i}\right) \mid t^i, t^{-i}\right)$. We assume that valuations are bounded. Furthermore, we assume quasi-linear utilities, i.e. utility is the valuation of an allocation minus the payment. We assume that agents' types are independently distributed. Let $\pi^i\left(t^i\right)$ denote the density on T^i. The joint density on T^{-i} is then given by

$$\pi^{-i}\left(t^{-i}\right) = \prod_{\substack{j \in N \\ j \neq i}} \pi^j\left(t^j\right).$$

Assume that agent i believes all other agents to report truthfully. If agent i has true type t^i, then his expected utility for making a report r^i is given by[1]

$$\begin{aligned} U^i(r^i \mid t^i) &= \int_{T^{-i}} \left(v^i\left(f\left(r^i, t^{-i}\right) \mid t^i, t^{-i}\right) - P_i\left(r^i, t^{-i}\right)\right) \pi^{-i}\left(t^{-i}\right) dt^{-i} \\ &= E_{-i}\left[v^i\left(f\left(r^i, t^{-i}\right) \mid t^i, t^{-i}\right) - P_i\left(r^i, t^{-i}\right)\right]. \end{aligned} \tag{1}$$

An allocation rule f is **Bayes-Nash incentive compatible** if there exists a payment rule P such that $\forall i \in N$ and $\forall r^i, \tilde{r}^i \in T^i$:

$$\begin{aligned} &E_{-i}\left[v^i\left(f\left(r^i, t^{-i}\right) \mid r^i, t^{-i}\right) - P_i\left(r^i, t^{-i}\right)\right] \\ &\geq E_{-i}\left[v^i\left(f\left(\tilde{r}^i, t^{-i}\right) \mid r^i, t^{-i}\right) - P_i\left(\tilde{r}^i, t^{-i}\right)\right]. \end{aligned} \tag{2}$$

The constraints of (2) imply that also

$$\begin{aligned} &E_{-i}\left[v^i\left(f\left(\tilde{r}^i, t^{-i}\right) \mid \tilde{r}^i, t^{-i}\right) - P_i\left(\tilde{r}^i, t^{-i}\right)\right] \\ &\geq E_{-i}\left[v^i\left(f\left(r^i, t^{-i}\right) \mid \tilde{r}^i, t^{-i}\right) - P_i\left(r^i, t^{-i}\right)\right]. \end{aligned} \tag{3}$$

[1] The definition of the utility in (1) is only given for the continuous case. In the discrete case the integral is replaced with a sum.

By adding (2) and (3) we get the **2-cycle inequality**[2]

$$
\begin{aligned}
& E_{-i}\left[v^i\left(f\left(r^i,t^{-i}\right) \mid r^i,t^{-i}\right)-v^i\left(f\left(\tilde{r}^i,t^{-i}\right) \mid r^i,t^{-i}\right)\right] \\
\geq\; & E_{-i}\left[v^i\left(f\left(r^i,t^{-i}\right) \mid \tilde{r}^i,t^{-i}\right)-v^i\left(f\left(\tilde{r}^i,t^{-i}\right) \mid \tilde{r}^i,t^{-i}\right)\right]. \qquad (4)
\end{aligned}
$$

That the 2-cycle inequality holds for every pair $r^i, \tilde{r}^i \in T^i$ is a necessary condition for Bayes-Nash incentive compatibility.

3 The Network

In order to see that the constraints in (2) have a natural network interpretation it is useful to rewrite (2) as follows:

$$
\begin{aligned}
& E_{-i}\left[P_i\left(r^i,t^{-i}\right)-P_i\left(\tilde{r}^i,t^{-i}\right)\right] \\
\leq\; & E_{-i}\left[v^i\left(f\left(r^i,t^{-i}\right) \mid r^i,t^{-i}\right)-v^i\left(f\left(\tilde{r}^i,t^{-i}\right) \mid r^i,t^{-i}\right)\right]. \qquad (5)
\end{aligned}
$$

For each agent we build a complete directed graph T_f^i. A node is associated with each type and a directed edge is put between each ordered pair of nodes. For agent i the length of an edge directed from $\tilde{r}^i$ to r^i is denoted $l^i(\tilde{r}^i, r^i)$ and defined as the **cost of manipulation**:

$$
l^i\left(\tilde{r}^i,r^i\right)=E_{-i}\left[v^i\left(f\left(r^i,t^{-i}\right) \mid r^i,t^{-i}\right)-v^i\left(f\left(\tilde{r}^i,t^{-i}\right) \mid r^i,t^{-i}\right)\right]. \qquad (6)
$$

Symmetrically, we associate an edge directed from r^i to $\tilde{r}^i$ with length

$$
l^i\left(r^i,\tilde{r}^i\right)=E_{-i}\left[v^i\left(f\left(\tilde{r}^i,t^{-i}\right) \mid \tilde{r}^i,t^{-i}\right)-v^i\left(f\left(r^i,t^{-i}\right) \mid \tilde{r}^i,t^{-i}\right)\right].
$$

Notice that the 2-cycle inequality can be written as

$$
l^i\left(\tilde{r}^i,r^i\right)+l^i\left(r^i,\tilde{r}^i\right) \geq 0 \qquad \forall \tilde{r}^i, r^i \in T^i.
$$

Theorem 1 *An allocation rule f is Bayes-Nash incentive compatible if and only if $\forall i \in N$, T_f^i has no negative, finite cycle.*

Proof(Adapted from Rochet [7].)
First let us assume that f is Bayes-Nash incentive compatible. Take some agent i and let $C=\left\{r_1^i,\ldots,r_m^i,r_{m+1}^i=r_1^i\right\}$ denote a finite cycle in T_f^i. Using (5) and (6) we get that $\forall j \in \{1,\ldots,m\}$

$$
E_{-i}\left[P_i\left(r_{j+1}^i,t^{-i}\right)-P_i\left(r_j^i,t^{-i}\right)\right] \leq l^i\left(r_j^i,r_{j+1}^i\right).
$$

Adding up these inequalities yields

$$
0 \leq \sum_{j=1}^{m} l^i\left(r_j^i,r_{j+1}^i\right),
$$

[2] Expected payments can be cancelled since we work under the assumption of independently distributed types.

so C has non-negative length.

Conversely, let us assume that there exists no negative, finite cycle in T^i_f, $\forall i \in N$. For each agent i we pick an arbitrary source node $r^i_1 \in T^i_f$ and define $\forall r^i \in T^i_f$

$$p^i\left(r^i\right) = \inf \sum_{j=1}^{m} l^i\left(r^i_j, r^i_{j+1}\right), \tag{7}$$

where the infimum is taken over all finite paths from r^i_1 to $r^i_{m+1} = r^i$. Note that we allow for the empty path. Absence of a negative, finite cycle implies that $p^i\left(r^i_1\right) = 0$. Furthermore, $\forall r^i \in T^i_f$ we have

$$p^i\left(r^i_1\right) \leq p^i\left(r^i\right) + l^i\left(r^i, r^i_1\right)$$

which implies that $p^i\left(r^i\right)$ is finite. We also have for every pair $\tilde{r}^i, r^i \in T^i_f$

$$p^i\left(r^i\right) \leq p^i\left(\tilde{r}^i\right) + l^i\left(\tilde{r}^i, r^i\right).$$

Thus, by setting[3] $P_i\left(r^i, t^{-i}\right) = p^i\left(r^i\right)$, $\forall t^{-i} \in T^{-i}$ we get

$$\begin{aligned} & E_{-i}\left[P_i\left(r^i, t^{-i}\right) - P_i\left(\tilde{r}^i, t^{-i}\right)\right] \\ \leq\ & E_{-i}\left[v^i\left(f\left(r^i, t^{-i}\right) \mid r^i, t^{-i}\right) - v^i\left(f\left(\tilde{r}^i, t^{-i}\right) \mid r^i, t^{-i}\right)\right]. \end{aligned}$$

□

4 One-Dimensional Types

In the one-dimensional case we have $T^i \subseteq I\!R$. Now, let us introduce the following condition for the costs of manipulation:

Definition 1 *The costs of manipulation are **decomposition monotone** if* $\forall \underline{r}^i, \bar{r}^i \in T^i$ *and* $\forall r^i \in T^i$ *such that* $r^i = \underline{r}^i + \alpha\left(\bar{r}^i - \underline{r}^i\right)$,$\alpha \in (0,1)$ *we have that*

$$l^i\left(\underline{r}^i, \bar{r}^i\right) \geq l^i\left(\underline{r}^i, r^i\right) + l^i\left(r^i, \bar{r}^i\right).$$

Theorem 2 *Suppose that the costs of manipulation are decomposition monotone, then:*
For all $i \in N$, T^i_f *has no negative, finite cycle if and only if* T^i_f *has no negative 2-cycle.*

Proof
Necessity of the absence of negative 2-cycles follows trivially. For the other direction let us assume that there is no negative 2-cycle in T^i_f, $\forall i \in N$. Let

[3] Note that we could also set $E_{-i}\left[P_i\left(r^i, t^{-i}\right)\right] = p^i\left(r^i\right) + c$ which would allow for a variety of payment rules that yield the same expected payments up to an additive constant.

$C = \{r_1^i, \ldots, r_m^i, r_{m+1}^i = r_1^i\}$ denote a finite cycle in T_f^i. Whenever an edge of C connects two non-neighboring nodes, we substitute this edge with a path connecting the same two nodes via edges that have the same direction and only connect neighboring nodes. By doing this we generate a new cycle $\tilde{C}$ that has the same nodes as C but consists only of edges between neighboring nodes, see for example Figure 1. The edge lengths satisfying decomposition monotonicity

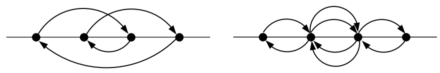

Fig. 1. Example for cycles C (left) and $\tilde{C}$ (right).

implies that the cycle length of C is larger or equal than the length of $\tilde{C}$. Since $\tilde{C}$ is a cycle, we know that at each node the number of edges entering equals the number of edges leaving. This implies that the length of $\tilde{C}$ can be written as the sum of lengths of 2-cycles. Since there are no negative 2-cycles, it follows that C has non-negative length.

□

An instance of the costs of manipulation are decomposition monotone is the case where the valuation function satisfies the following condition:

Definition 2 *Take $r^i, \tilde{r}^i, t^i, \tilde{t}^i \in T^i$ such that*

$$\begin{aligned} &E_{-i}\left[v^i\left(f\left(r^i, t^{-i}\right) \mid t^i, t^{-i}\right) - v^i\left(f\left(\tilde{r}^i, t^{-i}\right) \mid t^i, t^{-i}\right)\right] \\ \geq\ &E_{-i}\left[v^i\left(f\left(r^i, t^{-i}\right) \mid \tilde{t}^i, t^{-i}\right) - v^i\left(f\left(\tilde{r}^i, t^{-i}\right) \mid \tilde{t}^i, t^{-i}\right)\right]. \end{aligned}$$

The valuation function satisfies ***increasing expected differences*** *if $\forall \bar{t}^i \in T^i$ s.t. $\bar{t}^i = \tilde{t}^i + \alpha\left(t^i - \tilde{t}^i\right), \alpha > 1$ it holds that*

$$\begin{aligned} &E_{-i}\left[v^i\left(f\left(r^i, t^{-i}\right) \mid \bar{t}^i, t^{-i}\right) - v^i\left(f\left(\tilde{r}^i, t^{-i}\right) \mid \bar{t}^i, t^{-i}\right)\right] \\ \geq\ &E_{-i}\left[v^i\left(f\left(r^i, t^{-i}\right) \mid t^i, t^{-i}\right) - v^i\left(f\left(\tilde{r}^i, t^{-i}\right) \mid t^i, t^{-i}\right)\right]. \end{aligned}$$

In this condition we are dealing with the marginal change in expected valuation with respect to the report. Consider the change in expected valuation for making a report r^i instead of $\tilde{r}^i$. Assume that there exist types t^i and $\tilde{t}^i$ such that this change is larger or at least as large if the agent has true type t^i instead of $\tilde{t}^i$. Then we require that for all types which are even farther away from $\tilde{t}^i$ than t^i (in the direction of t^i) the change in expected valuation is at least as large as for t^i. This requirement is comparable to the condition known as increasing (or isotone) differences which asserts that the marginal change in valuation with respect to the allocation is increasing in the type.

Corollary 1 *Suppose that the valuation function satisfies increasing expected differences, then:*

For all $i \in N$, T_f^i has no negative, finite cycle if and only if T_f^i has no negative 2-cycle.

Proof
Necessity of the absence of negative 2-cycles follows trivially. For the other direction let us assume that there is no negative 2-cycle in T_f^i, $\forall i \in N$. Take any edge from T_f^i and denote its starting node $\underline{r}^i$ and its ending node $\bar{r}^i$. Suppose that there exists a $r^i \in T^i$ such that $r^i = \underline{r}^i + \alpha\left(\bar{r}^i - \underline{r}^i\right)$, $\alpha \in (0,1)$. Absence of negative 2-cycles implies

$$\begin{aligned}
&E_{-i}\left[v^i\left(f\left(r^i,t^{-i}\right) \mid r^i,t^{-i}\right) - v^i\left(f\left(\underline{r}^i,t^{-i}\right) \mid r^i,t^{-i}\right)\right] \\
\geq\ &E_{-i}\left[v^i\left(f\left(r^i,t^{-i}\right) \mid \underline{r}^i,t^{-i}\right) - v^i\left(f\left(\underline{r}^i,t^{-i}\right) \mid \underline{r}^i,t^{-i}\right)\right].
\end{aligned}$$

Since the valuation function satisfies increasing expected differences we have that

$$\begin{aligned}
&E_{-i}\left[v^i\left(f\left(r^i,t^{-i}\right) \mid \bar{r}^i,t^{-i}\right) - v^i\left(f\left(\underline{r}^i,t^{-i}\right) \mid \bar{r}^i,t^{-i}\right)\right] \\
\geq\ &E_{-i}\left[v^i\left(f\left(r^i,t^{-i}\right) \mid r^i,t^{-i}\right) - v^i\left(f\left(\underline{r}^i,t^{-i}\right) \mid r^i,t^{-i}\right)\right].
\end{aligned}$$

Adding on both sides of the inequality yields

$$\begin{aligned}
&E_{-i}\left[v^i\left(f\left(r^i,t^{-i}\right) \mid \bar{r}^i,t^{-i}\right) - v^i\left(f\left(\underline{r}^i,t^{-i}\right) \mid \bar{r}^i,t^{-i}\right)\right] \\
&+E_{-i}\left[v^i\left(f\left(\bar{r}^i,t^{-i}\right) \mid \bar{r}^i,t^{-i}\right) - v^i\left(f\left(r^i,t^{-i}\right) \mid \bar{r}^i,t^{-i}\right)\right] \\
\geq\ &E_{-i}\left[v^i\left(f\left(r^i,t^{-i}\right) \mid r^i,t^{-i}\right) - v^i\left(f\left(\underline{r}^i,t^{-i}\right) \mid r^i,t^{-i}\right)\right] \\
&+E_{-i}\left[v^i\left(f\left(\bar{r}^i,t^{-i}\right) \mid \bar{r}^i,t^{-i}\right) - v^i\left(f\left(r^i,t^{-i}\right) \mid \bar{r}^i,t^{-i}\right)\right].
\end{aligned}$$

Notice that the first and the last term on the lefthand side of the inequality cancel, hence

$$\begin{aligned}
&E_{-i}\left[v^i\left(f\left(\bar{r}^i,t^{-i}\right) \mid \bar{r}^i,t^{-i}\right) - v^i\left(f\left(\underline{r}^i,t^{-i}\right) \mid \bar{r}^i,t^{-i}\right)\right] \\
\geq\ &E_{-i}\left[v^i\left(f\left(r^i,t^{-i}\right) \mid r^i,t^{-i}\right) - v^i\left(f\left(\underline{r}^i,t^{-i}\right) \mid r^i,t^{-i}\right)\right] \\
&+E_{-i}\left[v^i\left(f\left(\bar{r}^i,t^{-i}\right) \mid \bar{r}^i,t^{-i}\right) - v^i\left(f\left(r^i,t^{-i}\right) \mid \bar{r}^i,t^{-i}\right)\right],
\end{aligned}$$

which can be written as

$$l^i\left(\underline{r}^i,\bar{r}^i\right) \geq l^i\left(\underline{r}^i,r^i\right) + l^i\left(r^i,\bar{r}^i\right).$$

So the costs of manipulation are decomposition monotone. The rest follows from the proof of Theorem 2.

□

5 Multi-Dimensional Types

In the multi-dimensional case we have $T^i \subseteq I\!R^k$.[4] We assume that all the T^i are convex. Furthermore, we now assume that an agent's valuation function is

[4] For the special case where the T^i are only lines in $I\!R^k$ the results of the foregoing section go through unchanged.

linear in his own true type. So if agent i has true type t^i and reports r^i while the others have true types t^{-i} and report r^{-i} then we can write agent i's valuation for the resulting allocation as

$$v^i\left(f\left(r^i, r^{-i}\right) \mid t^i, t^{-i}\right) = \alpha^i\left(f\left(r^i, r^{-i}\right) \mid t^{-i}\right) + \beta^i\left(f\left(r^i, r^{-i}\right) \mid t^{-i}\right) t^i.$$

Note that $\alpha^i : \Gamma \times T^{-i} \mapsto I\!R$ and $\beta^i : \Gamma \times T^{-i} \mapsto I\!R^k$, i.e. α^i assigns to every $\left(\gamma, t^{-i}\right) \in \Gamma \times T^{-i}$ a point in $I\!R$, whereas β^i assigns to every $\left(\gamma, t^{-i}\right) \in \Gamma \times T^{-i}$ a point in $I\!R^k$. Similarly, assuming agent i believes all other agents to report truthfully, we can write his expected valuation for reporting r^i while having true type t^i as

$$\begin{aligned} &E_{-i}\left[v^i\left(f\left(r^i, t^{-i}\right) \mid t^i, t^{-i}\right)\right] \\ &= E_{-i}\left[\alpha^i\left(f\left(r^i, t^{-i}\right) \mid t^{-i}\right)\right] + E_{-i}\left[\beta^i\left(f\left(r^i, t^{-i}\right) \mid t^{-i}\right)\right] t^i. \end{aligned} \tag{8}$$

Notice that such a linear valuation function satisfies increasing expected differences, see Definition 2. Also due to the linearity of the valuation function we have that

$$\frac{\partial E_{-i}\left[v^i\left(f\left(r^i, t^{-i}\right) \mid t^i, t^{-i}\right)\right]}{\partial t^i}\Big|_{t^i = r^i} = E_{-i}\left[\beta^i\left(f\left(r^i, t^{-i}\right) \mid t^{-i}\right)\right] \tag{9}$$

which is a vector field $T^i \mapsto I\!R^k$. A function φ: $T^i \mapsto I\!R$ is called a **potential function** for a vector field ψ: $T^i \mapsto I\!R^k$ if for any smooth path A joining $\underline{t}^i, \bar{t}^i \in T^i$

$$\int_A \psi = \varphi\left(\bar{t}^i\right) - \varphi\left(\underline{t}^i\right).$$

Theorem 3 *Suppose that $\forall i \in N$, T^i is convex and that agents have valuation functions that are linear w.r.t. their own true types then:*
For all $i \in N$, T_f^i has no negative, finite cycle if and only if
1) T_f^i has no negative 2-cycle and
2) $E_{-i}\left[\beta^i\left(f\left(r^i, t^{-i}\right) \mid t^{-i}\right)\right]$ has a potential function.

Proof
First let us assume that T_f^i has no negative, finite cycle. Necessity of the absence of negative 2-cycles follows trivially. Let $C = \left\{r_1^i, \ldots, r_m^i, r_{m+1}^i = r_1^i\right\}$ denote a finite cycle in T_f^i. Absence of negative, finite cycles implies that

$$\sum_{j=1}^{m} l^i\left(r_j^i, r_{j+1}^i\right) \geq 0$$

which can be rewritten using (6) and (8) as

$$\sum_{j=1}^{m} E_{-i}\left[\beta^i\left(f\left(r_{j+1}^i, t^{-i}\right) \mid t^{-i}\right) - \beta^i\left(f\left(r_j^i, t^{-i}\right) \mid t^{-i}\right)\right] r_{j+1}^i \geq 0.$$

This implies that

$$\sum_{j=1}^{m} E_{-i}\left[\beta^i\left(f\left(r_j^i, t^{-i}\right) \mid t^{-i}\right)\right]\left(r_{j+1}^i - r_j^i\right) \leq 0.$$

Thus, $E_{-i}\left[\beta^i\left(f\left(r^i, t^{-i}\right) \mid t^{-i}\right)\right]$ is cyclically monotone (see Rockafellar [8], p.238). This implies that there exists a convex function $\varphi: T^i \mapsto IR$ such that $E_{-i}\left[\beta^i\left(f\left(r^i, t^{-i}\right) \mid t^{-i}\right)\right]$ is a selection from its subdifferential mapping, i.e.

$$E_{-i}\left[\beta^i\left(f\left(r^i, t^{-i}\right) \mid t^{-i}\right)\right] \in \partial\varphi\left(r^i\right), \forall r^i \in T^i,$$

(see Rockafellar [8], Theorem 24.8). Furthermore, for any smooth path A joining $\underline{r}^i, \bar{r}^i \in T_f^i$ we have that

$$\int_A E_{-i}\left[\beta^i\left(f\left(r^i, t^{-i}\right) \mid t^{-i}\right)\right] = \varphi\left(\bar{r}^i\right) - \varphi\left(\underline{r}^i\right),$$

(see Krishna & Maenner [9], Theorem 1). So φ is a potential function for $E_{-i}\left[\beta^i\left(f\left(r^i, t^{-i}\right) \mid t^{-i}\right)\right]$.

For the converse let us assume $\forall i \in N$ that there is no negative 2-cycle in T_f^i and that $E_{-i}\left[\beta^i\left(f\left(r^i, t^{-i}\right) \mid t^{-i}\right)\right]$ has a potential function. Take any edge from T_f^i and denote its starting node $\underline{r}^i$ and its ending node $\bar{r}^i$. Let A denote the line segment between $\underline{r}^i$ and $\bar{r}^i$, i.e. $A = \left\{r^i \in T^i \mid r^i = \underline{r}^i + \alpha\left(\bar{r}^i - \underline{r}^i\right), \alpha \in [0,1]\right\}$. Now we substitute $l^i\left(\underline{r}^i, \bar{r}^i\right)$ with $l^i\left(\underline{r}^i, r^i\right) + l^i\left(r^i, \bar{r}^i\right)$, where $r^i \in A$ with $\alpha \in (0,1)$. As mentioned above the valuation function satisfies increasing expected differences. Together with the absence of negative 2-cycles this implies that

$$l^i\left(\underline{r}^i, \bar{r}^i\right) \geq l^i\left(\underline{r}^i, r^i\right) + l^i\left(r^i, \bar{r}^i\right), \tag{10}$$

which can be shown in the same way as in the proof of Corollary 1. By repeated substitution we generate a path $\tilde{A} = \left\{r_1^i = \underline{r}^i, \ldots, r_m^i, r_{m+1}^i = \bar{r}^i\right\}$, $r_j^i \in A$, $\forall j \in \{1, \ldots, m+1\}$. Then (10) implies that the original edge is at least as long as $\tilde{A}$, i.e.

$$l^i\left(\underline{r}^i, \bar{r}^i\right) \geq \sum_{j=1}^{m} l^i\left(r_j^i, r_{j+1}^i\right).$$

Using (6) we can write

$$\begin{aligned}
& \sum_{j=1}^{m} l^i\left(r_j^i, r_{j+1}^i\right) \\
= & \sum_{j=1}^{m} E_{-i}\left[v^i\left(f\left(r_{j+1}^i, t^{-i}\right) \mid r_{j+1}^i, t^{-i}\right) - v^i\left(f\left(r_j^i, t^{-i}\right) \mid r_{j+1}^i, t^{-i}\right)\right] \\
= & E_{-i}\left[v^i\left(f\left(\bar{r}^i, t^{-i}\right) \mid \bar{r}^i, t^{-i}\right) - v^i\left(f\left(\underline{r}^i, t^{-i}\right) \mid \underline{r}^i, t^{-i}\right)\right] \\
& - \sum_{j=1}^{m} E_{-i}\left[v^i\left(f\left(r_j^i, t^{-i}\right) \mid r_{j+1}^i, t^{-i}\right) - v^i\left(f\left(r_j^i, t^{-i}\right) \mid r_j^i, t^{-i}\right)\right].
\end{aligned}$$

By repeated substitution we can generate paths with more and more edges. In the limit the distance between neighboring nodes goes to zero. Therefore, by using (9), we have that the length of $\tilde{A}$ goes to

$$E_{-i}\left[v^i\left(f\left(\bar{r}^i,t^{-i}\right)\mid \bar{r}^i,t^{-i}\right)-v^i\left(f\left(\underline{r}^i,t^{-i}\right)\mid \underline{r}^i,t^{-i}\right)\right]-\int_A E_{-i}\left[\beta^i\left(f\left(r^i,t^{-i}\right)\mid t^{-i}\right)\right],$$

as $m \to \infty$. Since $E_{-i}\left[\beta^i\left(f\left(r^i,t^{-i}\right)\mid t^{-i}\right)\right]$ has a potential function we can write

$$\int_A E_{-i}\left[\beta^i\left(f\left(r^i,t^{-i}\right)\mid t^{-i}\right)\right]=\varphi\left(\bar{r}^i\right)-\varphi\left(\underline{r}^i\right),$$

where φ denotes the potential function. Thus, it follows that

$$\begin{aligned}&l^i\left(\underline{r}^i,\bar{r}^i\right)\\ &\geq E_{-i}\left[v^i\left(f\left(\bar{r}^i,t^{-i}\right)\mid \bar{r}^i,t^{-i}\right)-v^i\left(f\left(\underline{r}^i,t^{-i}\right)\mid \underline{r}^i,t^{-i}\right)\right]-\varphi\left(\bar{r}^i\right)+\varphi\left(\underline{r}^i\right).\end{aligned}$$

Now, let $C=\left\{r_1^i,\ldots,r_m^i,r_{m+1}^i=r_1^i\right\}$ denote a finite cycle in T_f^i. The result in (11) implies for the length of C that

$$\begin{aligned}&\sum_{j=1}^m l^i\left(r_j^i,r_{j+1}^i\right)\\ &\geq \sum_{j=1}^m E_{-i}\left[v^i\left(f\left(r_{j+1}^i,t^{-i}\right)\mid r_{j+1}^i,t^{-i}\right)-v^i\left(f\left(r_j^i,t^{-i}\right)\mid r_j^i,t^{-i}\right)\right]-\varphi\left(r_{j+1}^i\right)+\varphi\left(r_j^i\right)\\ &=0,\end{aligned}$$

so C has non-negative length.

$\square$

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